

A Fractional Acoustic Wave Equation from Multiple Relaxation Loss and Conservation Laws^{*}

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Abstract: This work concerns causal acoustical wave equations which imply frequency power-law attenuation. A connection between the five-parameter fractional Zener wave equation, which is derived from a fractional stress-strain relation plus conservations of mass and momentum, and the physically well established multiple relaxation framework is developed. It is shown that for a certain continuous distribution of relaxation mechanisms, the two descriptions are equal.

Keywords: Wave equations, attenuation, multiple relaxation, fractional modeling, power law descriptions, frequency dispersion.

1. INTRODUCTION

2. THEORY

The multiple relaxation mechanism framework of Nachman et al. (1990) is widely considered as adequate for acoustic wave modeling in lossy complex media like those encountered in medical ultrasound. It relies on thermodynamics and first principles of acoustical physics. The corresponding wave equation for N relaxation mechanisms is a causal partial differential equation with its highest time derivative order $N+2$. We denote this the Nachman–Smith–Waag (NSW) model.

Attenuation in complex media often follows a power law: $\alpha_k(\omega) \propto \omega^\eta$, with $\eta \in [0, 2]$ (Szabo and Wu (2000)). The range where experiments indicate this may cover many frequency decades. In order to make the NSW model attenuation adequately follow ω^η , either the valid frequency band must be narrow, or the number of assumed mechanisms N must be large thus inferring a partial differential equation of very high order.

Another way to derive a lossy wave equation is to combine the principles of mass and momentum conservation with some stress–strain relation. This constitutive relation may include fractional time-derivatives, exemplified by the fractional Zener model by Holm and Näsholm (2011). The resulting wave equation is causal and the corresponding attenuation follows power laws within wide frequency bands.

The purpose of the present work is to demonstrate the link between the NSW and the fractional Zener models via a continuum of relaxation mechanisms. Relevant parts of Näsholm and Holm (2011) are reviewed and reformulated. In addition a more general deduction is provided where the fractional Zener model parameters α and β are not necessarily equal. We aim to encourage the acoustical community to more frequently adopt fractional calculus descriptions for wave modeling in complex media.

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2.1 Conservation laws and Generalized Compressibility

The linearized conservation of mass corresponds to the strain being defined by

$$\epsilon(t) = \nabla u(x, t), \xleftrightarrow{\mathcal{F}} \epsilon(\omega) = -ik u(k, \omega), \quad (1)$$

where u is the displacement and the symbol \mathcal{F} denotes transformation into the spatio-temporal frequency domain where ω is the angular frequency and k the wavenumber.

The linearized conservation of momentum is expressed as

$$\nabla \sigma(t) = \rho_0 \frac{\partial^2 u(x, t)}{\partial t^2}, \xleftrightarrow{\mathcal{F}} -ik\sigma(\omega) = \rho_0 (i\omega)^2 u(k, \omega), \quad (2)$$

where ρ_0 is the steady-state mass density and σ denotes the stress, which in this context corresponds to the negative of the pressure.

The frequency-domain generalized compressibility is defined as the ratio between strain and stress: $\kappa(\omega) \triangleq \epsilon(\omega)/\sigma(\omega)$, therefore being related to the constitutive stress–strain relation. Combining this definition with the conservation laws (1) and (2) gives

$$k^2(\omega) = \omega^2 \rho_0 \kappa(\omega) \quad (3)$$

$$\xleftrightarrow{\mathcal{F}} \nabla^2 u(x, t) - \frac{d^2}{dt^2} [\kappa(t) * u(x, t)] = 0. \quad (4)$$

Under circumstances where the linearized conservations of mass and momentum are valid, the wave equation is thus completely determined by the generalized compressibility.

The generalized compressibility $\kappa(\omega)$ as given above is sometimes (e.g. in viscoelasticity) called complex compliance $J^*(\omega) = 1/G^*(\omega)$, where $G^*(\omega)$ is the complex modulus.

2.2 A Continuum of NSW Relaxation Processes

The NSW model of multiple discrete relaxation processes results in the generalized compressibility

$$\kappa(\omega) = \kappa_0 - i\omega \sum_{\nu=1}^N \frac{\kappa_\nu \tau_\nu}{1 + i\omega \tau_\nu}, \quad (5)$$

where the mechanisms $\nu = 1 \dots N$, have the relaxation times τ_1, \dots, τ_N and the compressibility contributions $\kappa_1, \dots, \kappa_N$ (Nachman et al. (1990)).

Following Näsholm and Holm (2011), a representation of (5) when considering a continuum of relaxation mechanisms distributed in the frequency band $\Omega \in [\Omega_1, \Omega_2]$ with the compressibility contributions described by the distribution $\kappa_\nu(\Omega)$ becomes

$$\kappa_N(\omega) \triangleq \kappa_0 - i\omega \int_{\Omega_1}^{\Omega_2} \frac{\kappa_\nu(\Omega)}{\Omega + i\omega} d\Omega. \quad (6)$$

Letting the limits of the integral go between $\Omega_1 = 0$ and $\Omega_2 = \infty$, and instead incorporating any possible relaxation distribution bandwidth limitation into $\kappa_\nu(\Omega)$, the integral above is a Stieltjes transform. Applying the Laplace transform relation

$$\mathcal{L}_\Omega^{-1} \left\{ \frac{1}{\Omega + i\omega} \right\} (t) = e^{-i\omega t}, \quad (7)$$

the generalized compressibility (6) becomes

$$\begin{aligned} \kappa_N(\omega) &= \kappa_0 - i\omega \int_0^\infty \kappa_\nu(\Omega) \int_0^\infty e^{-\Omega t} e^{-i\omega t} dt d\Omega \\ &= \kappa_0 - i\omega \mathcal{F}_t \left\{ H(t) \mathcal{L}_\Omega \{ \kappa_\nu(\Omega) \} (t) \right\} (\omega). \end{aligned} \quad (8)$$

2.3 The Fractional Zener Wave Equation

The five-parameter fractional Zener stress-strain constitutive relation is experimentally shown to be valid for a wide range of complex media, see the references in Holm and Näsholm (2011). As given by Bagley and Torvik (1983), it may be expressed as

$$\sigma(t) + \tau_\epsilon^\beta \frac{\partial^\beta \sigma(t)}{\partial t^\beta} = E_0 \left[\epsilon(t) + \tau_\sigma^\alpha \frac{\partial^\alpha \epsilon(t)}{\partial t^\alpha} \right]. \quad (9)$$

From this relation, the frequency-domain fractional Zener compressibility is obtained through the ratio $\epsilon(\omega)/\sigma(\omega)$:

$$\begin{aligned} \kappa_Z(\omega) &\triangleq \kappa_0 \frac{1 + (\tau_\epsilon i\omega)^\beta}{1 + (\tau_\sigma i\omega)^\alpha} \\ &= \kappa_0 - i\omega \kappa_0 \frac{(i\omega)^{\alpha-1} - (\tau_\epsilon^\beta / \tau_\sigma^\alpha) (i\omega)^{\beta-1}}{\tau_\sigma^{-\alpha} + (i\omega)^\alpha} \end{aligned} \quad (10)$$

Due to thermodynamic constraints, β is restricted to be smaller than or equal to α (Glöckle and Nonnenmacher (1991)).

Insertion of the generalized compressibility (10) into the dispersion relation (4), generates the time-domain fractional Zener wave equation (Holm and Näsholm (2011))

$$\nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \tau_\sigma^\alpha \frac{\partial^\alpha}{\partial t^\alpha} \nabla^2 u - \frac{\tau_\epsilon^\beta}{c_0^2} \frac{\partial^{\beta+2} u}{\partial t^{\beta+2}} = 0. \quad (11)$$

2.4 Connecting the NSW and the Fractional Zener Models

Provided that the conservations of mass (1) and momentum (2) are valid, and provided that the NSW generalized

compressibility $\kappa_N(\omega)$ of (8) is equal to the fractional Zener generalized compressibility $\kappa_Z(\omega)$ of (10), the dispersion relations from (3) are also equal. Because the dispersion relation is a spatio-temporal Fourier representation of the wave equation, $\kappa_N(\omega) = \kappa_Z(\omega)$ thus implies that the NSW wave equation becomes equal to the fractional Zener wave equation (11). Direct comparison of $\kappa_N(\omega)$ in (8) to $\kappa_Z(\omega)$ in (10), tells that they are equal in case the following is true:

$$\begin{aligned} \mathcal{F}_t \left\{ H(t) \mathcal{L}_\Omega \{ \kappa_\nu(\Omega) \} (t) \right\} (\omega) \\ = \kappa_0 \frac{(i\omega)^{\alpha-1} - (\tau_\epsilon^\beta / \tau_\sigma^\alpha) (i\omega)^{\alpha-(\alpha-\beta+1)}}{\tau_\sigma^{-\alpha} + (i\omega)^\alpha}. \end{aligned} \quad (12)$$

First we choose to study the case $\alpha = \beta$, which was also treated in Näsholm and Holm (2011). Inverse Fourier transformation of both sides of (12), then gives

$$\begin{aligned} H(t) \mathcal{L}_\Omega \{ \kappa_\nu(\Omega) \} (t) \\ = \kappa_0 (1 - \tau_\epsilon^\alpha / \tau_\sigma^\alpha) \mathcal{F}_\omega^{-1} \left\{ \frac{(i\omega)^{\alpha-1}}{\tau_\sigma^{-\alpha} + (i\omega)^\alpha} \right\} (t) \\ = \kappa_0 (1 - \tau_\epsilon^\alpha / \tau_\sigma^\alpha) H(t) E_{\alpha,1}(-t/\tau_\sigma^\alpha), \end{aligned} \quad (13)$$

where $E_{a,b}(\cdot)$ is the Mittag-Leffler function (see Appendix A), and $H(t)$ is the Heaviside step function. The Fourier transform relation used in the last step above is given in (A.2). Moreover, the inverse Laplace transform relation of (A.3), Eq. (13) hence gives

$$\begin{aligned} \kappa_\nu(\Omega) &= \kappa_0 (1 - \tau_\epsilon^\alpha / \tau_\sigma^\alpha) f_{\alpha,1}(\Omega, \tau_\sigma^{-\alpha}) \\ &= \frac{1}{\pi} \frac{\kappa_0 (\tau_\sigma^\alpha - \tau_\epsilon^\alpha) \Omega^{\alpha-1} \sin(\alpha\pi)}{(\tau_\sigma \Omega)^{2\alpha} + 2(\tau_\sigma \Omega)^\alpha \cos(\alpha\pi) + 1} \triangleq \kappa_{\nu\text{ML}}(\Omega) \end{aligned} \quad (14)$$

where $f_{\alpha,1}(\Omega, a)$ was inserted from (A.4).

For the more general case $\beta \leq \alpha$, inverse Fourier transform on both sides of (12) instead gives

$$\begin{aligned} H(t) \mathcal{L}_\Omega \{ \kappa_\nu(\Omega) \} (t) &= \\ \kappa_0 \mathcal{F}_\omega^{-1} \left\{ \frac{(i\omega)^{\alpha-1}}{\tau_\sigma^{-\alpha} + (i\omega)^\alpha} \right\} (t) \\ - \kappa_0 (\tau_\epsilon^\beta / \tau_\sigma^\alpha) \mathcal{F}_\omega^{-1} \left\{ \frac{(i\omega)^{\alpha-(\alpha-\beta+1)}}{\tau_\sigma^{-\alpha} + (i\omega)^\alpha} \right\} (t). \end{aligned} \quad (15)$$

Proceeding in a similar manner as for the $\alpha = \beta$ case then gives the distribution

$$\begin{aligned} \kappa_\nu(\Omega) &= \kappa_0 f_{\alpha,1}(\Omega, \tau_\sigma^{-\alpha}) - \kappa_0 (\tau_\epsilon^\beta / \tau_\sigma^\alpha) f_{\alpha, \alpha-\beta+1}(\Omega, \tau_\sigma^{-\alpha}) \\ &= \frac{\kappa_0 \tau_\sigma^\alpha}{\pi} \cdot \frac{\Omega^{\alpha-1} \sin(\alpha\pi)}{(\tau_\sigma \Omega)^{2\alpha} + 2(\tau_\sigma \Omega)^\alpha \cos(\alpha\pi) + 1} \\ &\quad - \frac{\kappa_0 \tau_\epsilon^\beta}{\pi} \cdot \frac{\Omega^{\beta-1} \sin(\beta\pi) - \tau_\sigma^\alpha \Omega^{\alpha-\beta+1} \sin((\alpha-\beta)\pi)}{(\tau_\sigma \Omega)^{2\alpha} + 2(\tau_\sigma \Omega)^\alpha \cos(\alpha\pi) + 1} \\ &\triangleq \kappa'_{\nu\text{ML}}(\Omega). \end{aligned} \quad (16)$$

We have thus shown that the fractional Zener wave equation (11) may be obtained within the Nachman-Smith-Waag framework of multiple relaxation (Nachman et al. (1990)), when assuming a continuum of relaxation mechanisms with the compressibility contribution as given by the distribution $\kappa'_{\nu\text{ML}}(\Omega)$ of (16).

In viscoelasticity, a relaxation time spectrum (below denoted $\tilde{H}(\tau)$) related to $\kappa_\nu(\Omega)$ is commonly studied (see e.g. Glöckle and Nonnenmacher (1991) and references therein). It is related to the complex modulus through

$$G(t) = G_\infty + \int_{-\infty}^{\infty} \tilde{H}(\tau) e^{-t/\tau} d \ln \tau. \quad (17)$$

It may be shown that for the 5-parameter fractional Zener model, when setting $\Omega = \tau^{-1}$, the τ -dependency of $\tilde{H}(\tau)$ differs by a factor τ to $\kappa_{\nu\text{ML}}(\Omega)$ of (16). Figs. 5 and 6 of Glöckle and Nonnenmacher (1991) illustrate that $\alpha = \beta$ gives symmetric $\tilde{H}(\tau)$, while $\alpha \neq \beta$ breaks the symmetry, most significantly far from the peak region. Such relaxation spectra are experimentally observed for complex media, e.g. natural rubber.

3. ATTENUATION AND PHASE VELOCITY EXAMPLES

The conventional decomposition of the frequency-dependent wavenumber into its real and imaginary parts, gives the phase velocity $c_p(\omega) = \omega/\Re\{k\}$ and attenuation $\alpha_k(\omega) = -\Im\{k\}$.

In general, the attenuation and the phase velocity are thus given from the dispersion relation (3) as

$$\begin{aligned} \alpha_k(\omega) &= -\Im\{k\} = -\omega\sqrt{\rho_0}\Im\left\{\sqrt{\kappa(\omega)}\right\} \quad \text{and} \\ c_p(\omega) &= \omega/\Re\{k\} = \rho_0^{-1/2}/\Re\left\{\sqrt{\kappa(\omega)}\right\}. \end{aligned} \quad (18)$$

For the fractional Zener wave equation, this results in three distinct regions with attenuation power-laws (Näsholm and Holm (2011)): $\alpha_k \propto \omega^{1+\alpha}$ in a low-frequency regime, $\alpha_k \propto \omega^{1-\alpha/2}$ in an intermediate frequency regime, and $\alpha_k \propto \omega^{1-\alpha}$ in a high-frequency regime.

In the following, the fractional Zener phase velocities and attenuations are further investigated numerically for the $\alpha = \beta$ case in a similar manner as in Näsholm and Holm (2011). This is done explicitly by insertion of $\kappa(\omega) = \kappa_N(\omega)$ into (18). The results from such calculations are compared to what is found by insertion of the distribution $\kappa_{\nu\text{ML}}(\Omega)$ of (14) into the NSW generalized compressibility integral formula (6). This generalized compressibility is finally applied to (18), from which $\alpha_k(\omega)$ and $c_p(\omega)$ are found.

We use the latter calculation method to explore the effect of letting the continuum of relaxation mechanisms populate only a bounded frequency interval, rather than the entire $\Omega \in [0, \infty]$ region.

Figure 1 compares attenuation curves and shows the distributions $\kappa_\nu(\Omega)$, while Fig. 2 displays the corresponding frequency-dependent phase velocity. Note the high-frequency asymptote of $\kappa_\nu(\Omega)$, which has the fractal property of being proportional to $\Omega^{-\alpha-1}$. The integral over $\Omega \in [\Omega_1, \Omega_2]$ in the calculation of $\kappa_N(\omega)$ from (6) is evaluated numerically using the recursive adaptive Simpson quadrature method.

4. CONCLUSION

This work shows analytically that the lossy fractional Zener wave equation (9) (Holm and Näsholm (2011)) may be attained within the NSW multiple relaxation loss framework (Nachman et al. (1990)), given that a continuum of relaxation processes are weighted appropriately following $\kappa'_{\nu\text{ML}}(\Omega)$ as described in (16). The result may be seen as a generalization of what was presented in Näsholm

and Holm (2011) as the developments here also cover the case of the fractional Zener constitutive relation not having equal derivative orders α and β .

The advantages of the fractional Zener model is that it fits measurements well and that it is characterized by a small number of parameters, while the NSW model is more intuitive as it does not comprise fractional derivatives. It is also better rooted in fundamental physics.

REFERENCES

- Bagley, R.L. and Torvik, P.J. (1983). Fractional calculus — A different approach to the analysis of viscoelastically damped structures. *AIAA J.*, 21(5), 741–748.
- Djrbashian, M.M. (1966). *Integral transforms and representations of functions in the complex domain*, chapter 3–4. Nauka, Moscow, USSR. In Russian.
- Djrbashian, M.M. (1993). *Harmonic analysis and boundary value problems in the complex domain*, chapter 1. Birkhäuser, Basel, Switzerland.
- Glöckle, W.G. and Nonnenmacher, T.F. (1991). Fractional integral operators and Fox functions in the theory of viscoelasticity. *Macromolecules*, 24(24), 6426–6434.
- Haubold, H.J., Mathai, A.M., and Saxena, R.K. (2011). Mittag-Leffler functions and their applications. *Journal of Applied Mathematics*, 2011, 1–51.
- Holm, S. and Näsholm, S.P. (2011). A causal and fractional all-frequency wave equation for lossy media. *J. Acoust. Soc. Am.*, 130(4), 2195–2202.
- Mittag-Leffler, M.G. (1903). Sur la nouvelle fonction $E_\alpha(x)$ (On the new function $E_\alpha(x)$). *C. R. Acad. Sci. Paris*, 137, 554–558.
- Nachman, A.I., Smith III, J.F., and Waag, R.C. (1990). An equation for acoustic propagation in inhomogeneous media with relaxation losses. *J. Acoust. Soc. Am.*, 88, 1584–1595.
- Näsholm, S.P. and Holm, S. (2011). Linking multiple relaxation, power-law attenuation, and fractional wave equations. *J. Acoust. Soc. Am.*, 130(5), 3038–3045.
- Podlubny, I. (1999). *Fractional differential equations*, chapter 1–2. Academic Press, New York.
- Szabo, T.L. and Wu, J. (2000). A model for longitudinal and shear wave propagation in viscoelastic media. *J. Acoust. Soc. Am.*, 107, 2437–2446.
- Wiman, A. (1905). Über den Fundamentalsatz in der Theorie der Funktionen $E_\alpha(x)$ (About the fundamental theorem in the theory of the function $E_\alpha(x)$). *Acta Mathematica*, 29, 191–201.

Appendix A. THE MITTAG-LEFFLER FUNCTION

A.1 Definition and Fourier Transform Relation

The one-parameter Mittag-Leffler function was introduced in Mittag-Leffler (1903). A two-parameter analogy was presented in Wiman (1905), which may be written as

$$E_{a,b}(t) \triangleq \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(an+b)}, \quad (A.1)$$

where Γ is the Euler Gamma function and the parameters are commonly restricted to $\{a,b\} \in \mathbb{C}$, $\Re\{a,b\} > 0$, and $t \in \mathbb{C}$. See Haubold et al. (2011) for a comprehensive review of Mittag-Leffler function properties.

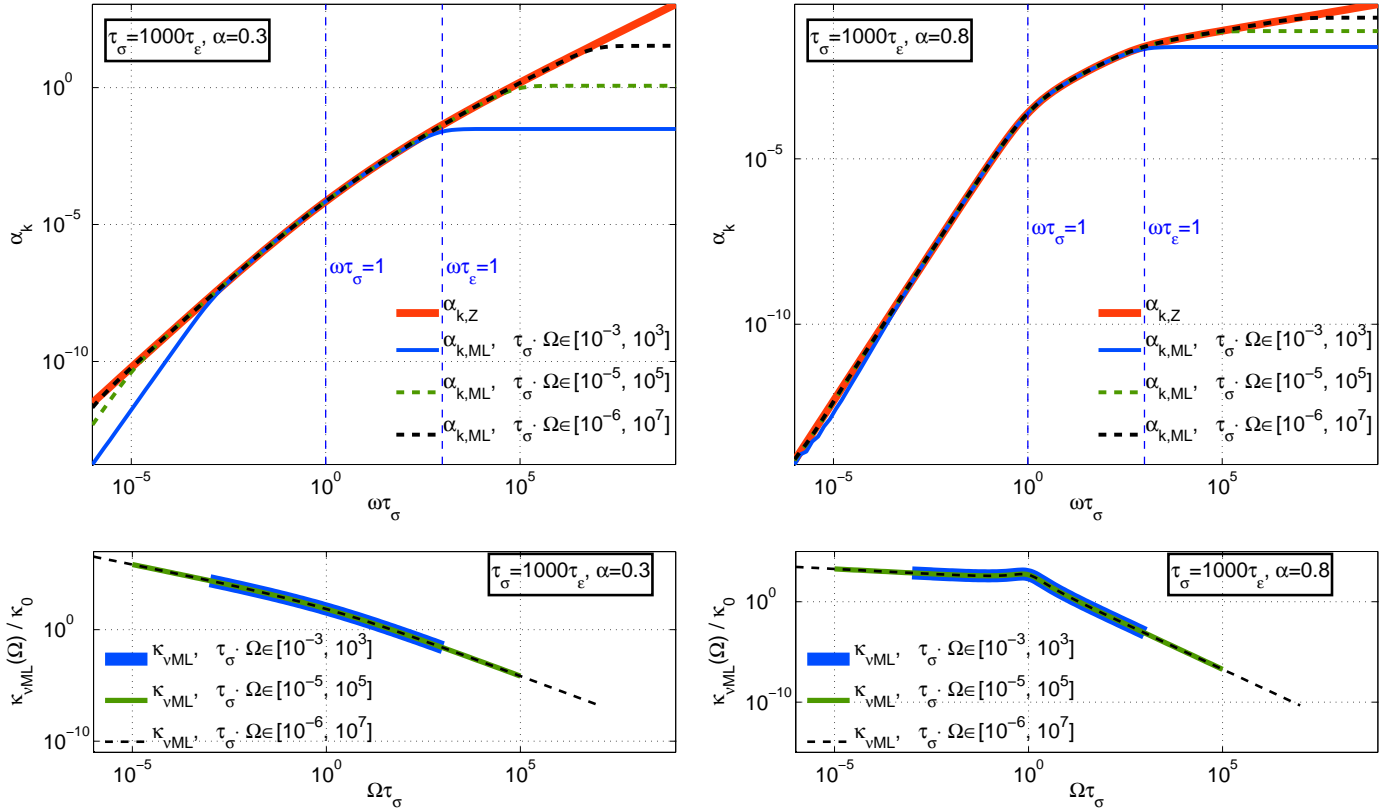


Fig. 1. Top panes: frequency-dependent attenuation for $\tau_\sigma = 1000\tau_\epsilon$ with the fractional derivative orders $\alpha = 0.3$ (top-left pane), and $\alpha = 0.8$ (top-right pane). The attenuation curves display both explicit calculations from the fractional Zener model, and calculations using the distribution $\kappa_{\nu\text{ML}}(\Omega)$ in the NSW model. Different choices of integration limits for Ω_1 and Ω_2 in (6) are exploited, as displayed in the legends. The horizontal axis represents normalized frequency. For visualization convenience, each attenuation curve is normalized to $\alpha_k = 1$ at $\omega\tau_\sigma = 1$. Bottom panes: the corresponding normalized effective compressibilities $\kappa_{\nu\text{ML}}(\Omega)$ of the continuum of relaxation processes as a function of normalized relaxation frequency $\Omega \cdot \tau_\sigma$ for $\alpha = 0.3$ (bottom-left pane), and $\alpha = 0.8$ (bottom-right pane). The Ω integration limits are given in the legends.

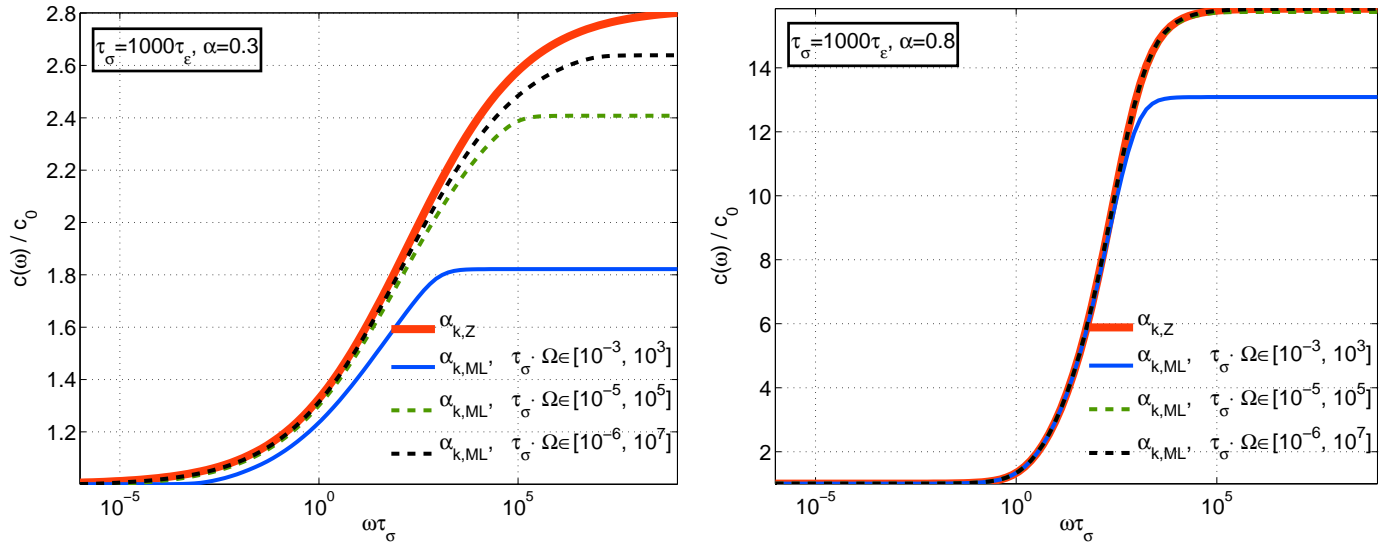


Fig. 2. Frequency-dependent phase velocity for $\tau_\sigma = 1000\tau_\epsilon$ with the fractional derivative orders $\alpha = 0.3$ (left pane), and $\alpha = 0.8$ (right pane). The curves display $c_p(\omega)$ as predicted by the fractional Zener model, as well as predicted by using the distribution $\kappa_{\nu\text{ML}}(\Omega)$ in the NSW model. Different choices of integration limits for Ω_1 and Ω_2 in (6) are exploited, as displayed in the legends. The horizontal axis represents normalized frequency.

A useful Fourier transform pair involving the Mittag-Leffler function is (Podlubny (1999))

$$\mathcal{F}\{H(t)t^{b-1}E_{a,b}(-At^a)\}(\omega) = \frac{(i\omega)^{a-b}}{A + (i\omega)^a}. \quad (\text{A.2})$$

A.2 Laplace Transform Integral Representation

The function $t^{b-1}E_{a,b}(-At^a)$ may for $0 < a \leq 1$ be written on an integral form (Djrbashian (1966, 1993)):

$$t^{b-1}E_{a,b}(-At^a) = \int_0^\infty e^{-\Omega t} f_{a,b}(\Omega, A) d\Omega, \quad (\text{A.3})$$

where

$$f_{a,b}(\Omega, A) = \frac{\Omega^{a-b}}{\pi} \frac{A \sin[(b-a)\pi] + \Omega^a \sin(b\pi)}{\Omega^{2a} + 2A\Omega^a \cos(a\pi) + A^2}. \quad (\text{A.4})$$